Math 125 - Summer 2019
Exam 2
Aug 8, 2019

Name: ______________________

Section: ______________________

Student ID Number: ______________________

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- There are 5 questions spanning 5 pages. Make sure your exam contains all these questions.
- You are allowed to use the Ti-30x IIS scientific calculator (**no graphing calculators**) and one **hand-written** 8.5 by 11 inch page (front and back) of notes.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- There is a **TABLE OF INTEGRALS ON THE BACK** of the exam, please tear it off and use it as a reference.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 60 minutes to complete the exam. Budget your time wisely.
SPEND NO MORE THAN 10-15 MINUTES PER PAGE!

GOOD LUCK!

1. Evaluate the integrals:

(a) $\displaystyle\int \frac{1}{x\cos^2(\ln x)}\,dx$

Solution Let $u = \ln x$, then $du = \frac{dx}{x}$ and we have

$$\begin{aligned}\int \frac{1}{x\cos^2(\ln x)}\,dx &= \int \frac{1}{\cos^2(u)}\,du \\ &= \int \sec^2 u\,du \\ &= \tan u + C = \boxed{\tan(\ln x) + C}\end{aligned}$$

(b) $\displaystyle\int \frac{x}{x^2-4x+3}\,dx$

Solution Factor the bottom yields $(x-1)(x-3)$, then we have:

$$\begin{aligned}\int \frac{x}{x^2-4x+3}\,dx &= \frac{3}{2}\int \frac{dx}{x-3} - \frac{1}{2}\int \frac{dx}{x-1} \\ &= \boxed{\frac{3}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C}\end{aligned}$$

2. Evaluate the integrals:

(a) $\displaystyle\int \frac{2}{4\sqrt{x}+x\sqrt{x}}\,dx$

Let $x = u^2$, so $dx = 2u\,du$, and the integral is

$$\begin{aligned}\int \frac{4u\,du}{4u+u^3} &= \int \frac{4\,du}{4+u^2}\\ &= \int \frac{du}{1+(u/2)^2}\\ &= 2\tan^{-1}(u/2)+C\\ &= \boxed{2\tan^{-1}(\sqrt{x}/2)+C}\end{aligned}$$

(b) $\displaystyle\int \frac{9}{t^3\sqrt{t^2-9}}\,dt$

Let $t = 3\sec(\theta)$ so $dt = 3\sec(\theta)\tan(\theta)\,d\theta$. Also $t^2-9 = 9\tan^2(\theta)$. This integral becomes

$$\begin{aligned}\int \frac{27\sec\theta\tan\theta\,d\theta}{27\sec^3(\theta)\cdot 3\tan(\theta)} &= \frac{1}{3}\int \frac{d\theta}{\sec^2(\theta)}\\ &= \frac{1}{3}\int \cos^2(\theta)\,d\theta\\ &= \frac{1}{3}\int \frac{1}{2}(1+\cos(2\theta))\,d\theta\\ &= \frac{1}{6}(\theta+\sin(2\theta)/2)+C\\ &= \frac{1}{6}(\cos^{-1}(3/t)+\sin(\theta)\cos(\theta))+C\end{aligned}$$

Drawing the appropriate triangle yields $\sin\theta = \frac{\sqrt{t^2-9}}{t}$ and $\cos\theta = 3/t$. Thus, our integral is

$$\boxed{\frac{1}{6}\cos^{-1}(3/t)+\frac{3\sqrt{t^2-9}}{2t^2}+C}$$

3. (a) Write the defintion of the average value of a function $f(x)$ over an interval $[a, b]$.

Solution

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

(b) Find a positive b so that the average value of $f(x) = 3x^2 - 18x$ on the interval $[0, b]$ is equal to 36.

Solution The question is asking us to consider when

$$\frac{1}{b}\int_0^b 3x^2 - 18x\,dx = 36.$$

We evaluate the left-hand-side to get

$$\frac{1}{b}\int_0^b 3x^2 - 18x\,dx = \frac{1}{b}\left(x^3 - 9x^2\right)\Big|_{x=0}^{b} = \frac{1}{b}\left(b^3 - 9b^2\right) = b^2 - 9b.$$

Now we set this equal to the right hand side and solve for b:

$$b^2 - 9b = 36 \implies b^2 - 9b - 36 = 0 \implies (b-12)(b+3) = 0.$$

Since the problem is asking for a postive b, then we ignore the negative option and thus our answer is $\boxed{b = 12.}$

4. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = e^x$ and $x = 1$. Find the volume of the solid obtained by revolving $\mathcal{R}$ about the y axis.

Solution: Using the shell method, the radius of a cylindrical shell we get from revolving a vertical slice at x from the axis of rotation is just x. The height of such a shell is e^x. Thus, the volume is

$$\int_0^1 2\pi x e^x \, dx = 2\pi \int_0^1 x e^x \, dx$$

To evaluate this, we use integration by parts with

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x. \end{aligned}$$

Thus, we get

$$\begin{aligned} 2\pi \int_0^1 x e^x \, dx &= 2\pi \left(x e^x \Big|_0^1 - \int_0^1 e^x \, dx \right) \\ &= 2\pi \left(e - e^x \Big|_0^1 \right) \\ &= 2\pi \Big(e - (e-1) \Big) \\ &= \boxed{2\pi} \end{aligned}$$

Washers are the wrong way to go about this, but if we do washers, the right and left functions change. Below the pink line at $y = 1$, the right boundary of $\mathcal{R}$ is the blue line $x = 1$, and the left boundary is the green line $x = 0$, the axis of rotation. So below the pink line $y = 1$ our washers are just disks, contributing $\int_0^1 \pi \cdot 1^2 \, dy = \pi$ to the total volume.

Above the pink line $y = 1$, the right boundary is $x = 1$, and the left boundary is the red curve $y = e^x$, the same as $x = \ln y$. This part extends from $y = 1$ to the intersection of the two curves, which happens at $y = e$. The volume we get from revolving this part is

$$\int_1^e \pi(1)^2 - \pi(\ln y)^2 \, dy = \pi(e-1) - \pi \int_1^e (\ln y)^2 \, dy$$

To compute this remaining integral, we do integration by parts with

$$\begin{aligned} u &= (\ln y)^2 & dv &= dy \\ du &= 2\frac{\ln y}{y} & v &= y. \end{aligned}$$

Thus, we get

$$\begin{aligned} \int_1^e (\ln y)^2 \, dy &= y(\ln y)^2 \Big|_1^e - \int_1^e 2y \frac{\ln y}{y} \, dy \\ &= [e(\ln e)^2 - 1(\ln 1)^2] - \int_1^e 2 \ln y \, dy \\ &= e - \int_1^e 2 \ln y \, dy \end{aligned}$$

To evaluate this last integral, we do integration by parts again, with

$$u = 2\ln y \qquad\qquad dv = dy$$
$$du = \frac{2}{y} \qquad\qquad v = y.$$

Thus, we get

$$\int_1^e 2\ln y\,dy = 2y\ln y\Big|_1^e - \int_1^e y\frac{2}{y} = 2e - \int_1^e 2\,dy = 2e - 2(e-1) = 2.$$

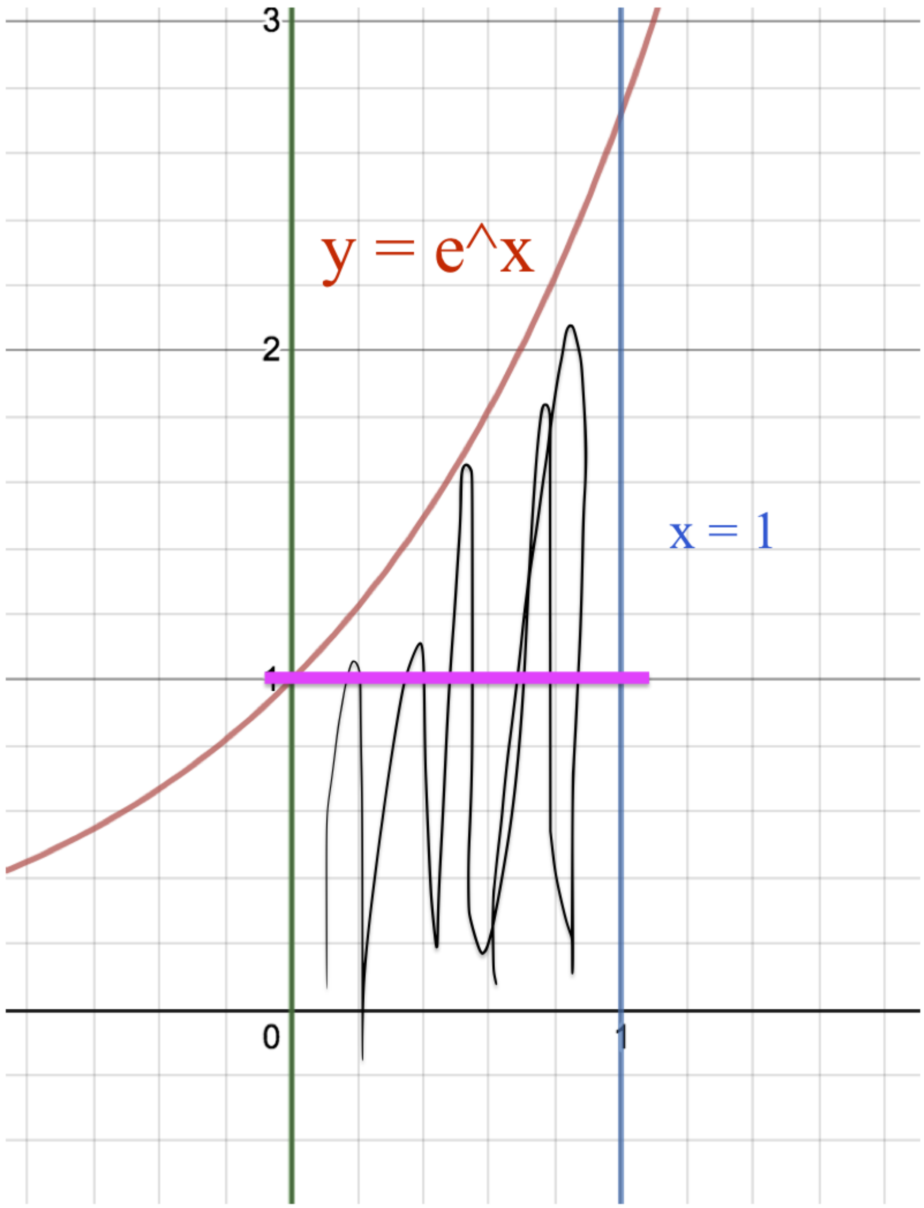
Plugging in, we get

$$\int_1^e (\ln y)^2\,dy = e - \int_1^e 2\ln y\,dy = e - 2.$$

Plugging in again, we get

$$\int_1^e \pi(1)^2 - \pi(\ln y)^2\,dy = \pi(e-1) - \pi\int_1^e (\ln y)^2\,dy = \pi(e-1) - \pi(e-2) = \pi(e-1-(e-2)) = \pi$$

Thus the total volume is $\pi + \pi = \boxed{2\pi}$


3
y = e^x
2
x = 1
1
0
1

5. Kobe Bryant is preparing to come out of retirement to compete for one more championship with the Los Angeles Lakers. In preparation for this Kobe pumps water out of a tank. The tank is formed by portion of the graphs below $y = 2$ and above $y = \sqrt{x}$ between $x = 0$ and $x = 4$ rotated around the y-axis. (Distances are in feet and water weighs 62.5 lb/ft^3.)

   Calculate the work required by Kobe to pump all the water out of the tank.

   Solution:

   Using that that work is force multiplied by distance we first seek to find the distance it takes for a slice of water that is y units up to be pumped to the top. This distance is $2 - y$ since the top of the tank is 2 feet high.

   Next we find the force needed. Since force is measured in pounds and we know the weight of the water is 62.5 lb/ft^3, then we actually aim to find the volume to make the units work.

   Since we are revolving around the y-axis we use dy to calculate the volume. Thus a typical slice will be a washer. The length from the y-axis to the right edge of the washer is the radius of the slice. Using that the right edge is given by the graph $y = \sqrt{x}$, then $x = y^2$ is the radius. So the volume of a typical slice will be $\pi(y^2)^2\Delta y = \pi y^4 \Delta y$.

   Hence, the total work is given by the integral is given by multiplying what we found above:

$$
\begin{aligned}
W &= 62.5 \int_0^2 \pi y^4 (2 - y)\, dy = 62.5\pi \int_0^2 2y^4 - y^5\, dy \\
&= 62.5\pi \left(\frac{2}{5} y^5 - \frac{1}{6} y^6 \right) \Big|_{y=0}^{2} \\
&= 62.5\pi \left(\frac{2^6}{5} - \frac{2^6}{6} \right) = 62.5\pi \left(\frac{6 \cdot 2^6}{30} - \frac{5 \cdot 2^6}{30} \right) \\
&= \frac{62.5 \cdot 2^6 \pi}{30} \approx 418.87 \text{ ft-lbs}
\end{aligned}
$$

Integration Table

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$
$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$	$\int b^x = \frac{1}{\ln(b)}b^x + C$
$\int \cos(ax)\,dx = \frac{1}{a}\sin(x) + C$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(x) + C$
$\int \sec^2(x)\,dx = \tan(x) + C$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\int \sec(x)\tan(x)\,dx = \sec(x) + C$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	$\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$
$\int \sec(x)\,dx = \ln\lvert\sec(x)+\tan(x)\rvert + C$	$\int \csc(x)\,dx = \ln\lvert\csc(x)-\cot(x)\rvert + C$

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\lvert\sec(x)+\tan(x)\rvert + C$$